A mixed problem for a Boussinesq hyperbolic equation with integral condition

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A MIXED PROBLEM FOR A BOUSSINESQ HYPERBOLIC EQUATION WITH INTEGRAL CONDITION

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Abstract. A hyperbolic problem wich combines a classical(Dirichlet) and a non-local contraint is considered.The existence and uniqueness of strong solutions are proved,we use a functionnal analysis method based on a priori estimate and on the density of the range of the operator generated by the considered problem.

1.INTRODUCTION

The first study of evolution problems with a nonlocal condition - the so called energy specification - goes back to Cannon[5] ,1963 Using an integral condition ,we proved the existence and uniqueness of the solution of a mixed problem wich combine a classical (Dirichlet)and an integral condition for the equation . Problems involving local and integral condition for hyperbolic equations are investigated by the energy inequalities method in [1] , [6] , [7] , [8] , [9] , [10] , [11], [12] .In this paper ,we prove the existence and uniqueness of the solution for the mixed problem $(1)-(5)$ .Our proof is based on a priori estimate and on the fact that the range of the operator generated by the considered problem is dense.

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2.Formulation of the problem

In the region $Q=(0,l)\times(0,T)$, with $l<\infty$ and $T<\infty$, we shall consider the problem

$$(1)\quad Lu=u_{tt}-(b(x,t)u_x)_x-\beta\frac{\partial^4 u}{\partial t^2\partial x^2}=f(x,t),\forall(x,t)\in Q$$

$$(2)\qquad l_1u=u\left(x,0\right)=\varphi_1\left(x\right),\quad x\in(0,l)$$

$$(3)\qquad l_2u=u_t\left(x,0\right)=\varphi_2\left(x\right),\quad x\in(0,l)$$

$$(4)\qquad u\left(0,t\right)=0,\qquad t\in(0,T)$$

$$(5)\qquad \int_0^l xu\left(x,t\right)dx=0,\quad t\in(0,T)$$

where $\beta\in IR_+^*$ and $b(x,t)$ and its derivatives satisfy the conditions:

C$_1$: $b_0\leq b(x,t)\leq b_1$, $b_t\left(x,t\right)\leq b_2$, $b_x\left(x,t\right)\leq b_3$,
for any $(x,t)\in\overline{Q}$,
C$_2$: $b_{tt}\left(x,t\right)\leq b_4$, $b_{xt}\left(x,t\right)\leq b_5$,for any $(x,t)\in\overline{Q}$.

The functions f, φ_1 and φ_2 are known functions wich satisfy the compatibility conditions:

$\varphi_1\left(0\right)=\varphi_2\left(0\right)=\int_0^l x\varphi_1\left(x\right)dx=\int_0^l x\varphi_2\left(x\right)dx=0.$

3. functional Spaces

The problem (1)-(5) can be put in the following operator form: $Lu = \mathcal{F}, \quad u \in D(L)$, where:

$$Lu = (\mathcal{L}u, l_1 u, l_2 u) \text{ and } \mathcal{F} = (f, \varphi_1, \varphi_2).$$

The operator L is considered from B to H,where B is the Banach space consisting of functions $u \in L^2(Q)$, satisfying conditions (4)and(5) with the finite norm:

$$\|u\|_B^2 = \sup_{0 \le \tau \le T} \left[\|u(.,\tau)\|^2_{L^2(0,l)} + \|u_t(.,\tau)\|^2_{L^2(0,l)} \right]$$

and F is the Hilbert space $L^2(Q) \times L^2(0,l) \times L^2(0,l)$ equipped with the norm:

$$\|\mathcal{F}\|_H^2 = \|f\|^2_{L^2(Q^\tau)} + \|\varphi_1\|^2_{L^2(0,l)} + \|\varphi_2\|^2_{L^2(0,l)}.$$

Let $D(L)$ denote the domain of L ,which is the set of all functions $u \in L^2(Q)$ for which $u_t, u_x, u_{tx}, u_{tt}, u_{ttx} \in L^2(Q)$ and satisfying conditions(4)and(5).

4. A priori estimate and its consequences

Theorem 1: For any function $u \in D(L)$ satisfyies conditions C_1-C_2 there exists a positive constant c, such that

$$\|u\|_B \le c \|Lu\|_H \,, \tag{7}$$

Proof : We consider the scalair product in $L^2(Q^\tau)$ of the operator $\mathcal{L}u$ and Mu, where $Mu = x\Im_x^* u_t - \Im x^*(\rho u_t)$, with $Q^\tau = (0,l) \times (0,\tau)$, $0 \le \tau \le T$, and $\Im_x^* v = \int_x^l v(\xi,t)d\xi$, we obtain

(8)
$$
\begin{aligned}
(\mathcal{L}u, Mu)_{L^2(Q^\tau)} &= (u_{tt}, x\Im_x^* u_t)_{L^2(Q^\tau)} - ((b\,(x,t)\,u_x)_x\,, x\Im_x^* u_t)_{L^2(Q^\tau)} \\
&\quad - \beta\,(u_{ttxx}, x\Im_x^* u_t)_{L^2(Q^\tau)} - (u_{tt}, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} \\
&\quad + ((b\,(x,t)\,u_x)_x\,, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} + \beta\,(u_{ttxx}, \Im_x^*(\rho u_t))_{L^2(Q^\tau)}\,.
\end{aligned}
$$

Making use of conditions (2)-(5)and integrating by parts we estabilish the equalities:

$$
\begin{aligned}
(u_{tt}, x\Im_x^* u_t)_{L^2(Q^\tau)} &= \frac{1}{2}\left\|\Im_x^* u_t(.,\tau)\right\|_{L^2(0,l)}^2 \\
-\frac{1}{2}\left\|\Im_x^*\varphi_2\right\|_{L^2(0,l)}^2 &- (\Im_x^* u_{tt}, u_t)_{L^2_\rho(Q^\tau)}\,,
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&- ((b\,(x,t)\,u_x)_x\,, x\Im_x^*(u_t))_{L^2(Q^\tau)} \\
&= \frac{1}{2}\left\|\sqrt{b\,(.,\tau)}u(.,\tau)\right\|_{L^2(0,l)}^2 - \frac{1}{2}\left\|\sqrt{b\,(.,0)}\varphi_1\right\|_{L^2(0,l)}^2 \\
&\quad -\frac{1}{2}\left\|\sqrt{b_t\,(.,t)}u\right\|_{L^2(Q^\tau)}^2 - (b_x\,(x,t)\,u, \Im_x^* u_t)_{L^2(Q^\tau)} \\
&\quad - (b\,(x,t)\,u_x, u_t)_{L^2_\rho(Q^\tau)}\,,
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
-\beta\,(u_{ttxx}, x\Im_x^*(u_t))_{L^2(Q^\tau)} &= \frac{\beta}{2}\left\|u_t(.,\tau)\right\|_{L^2(0,l)}^2 \\
-\frac{\beta}{2}\left\|\varphi_2\right\|_{L^2(0,l)}^2 &- \beta\,(u_{ttx}, u_t)_{L^2_\rho(Q^\tau)}\,.
\end{aligned}
\tag{11}
$$

$$
- (u_{tt}, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} = (\Im_x^* u_{tt}, u_t)_{L^2_\rho(Q^\tau)}\,,
\tag{12}
$$

$$
((b\,(x,t)\,u_x)_x\,, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} = (b\,(x,t)\,u_x, u_t)_{L^2_\rho(Q^\tau)}\,,
\tag{13}
$$

$$
\beta\,(u_{ttxx}, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} = \beta\,(u_{ttx}, u_t)_{L^2_\rho(Q^\tau)}\,.
\tag{14}
$$

Combining equalities (9)-(14) and(8)we obtain :

$$\begin{aligned}(15)\qquad &\frac{1}{2}\left\|\Im_x^* u_t(.,\tau)\right\|_{L^2(0,l)}^2+\frac{1}{2}\left\|\sqrt{b\,(.,t)}u(.,\tau)\right\|_{L^2(0,l)}^2\\&\qquad+\frac{\beta}{2}\left\|u_t(.,\tau)\right\|_{L^2(0,l)}^2\\&=\frac{1}{2}\left\|\Im_x^*\varphi_2\right\|_{L^2(0,l)}^2+\frac{1}{2}\left\|\sqrt{b\,(.,t)}\varphi_1\right\|_{L^2(0,l)}^2\\&+\frac{\beta}{2}\left\|\varphi_2\right\|_{L^2(0,l)}^2+\frac{1}{2}\left\|\sqrt{b_t}u\right\|_{L^2(Q^\tau)}^2+\left(b_x\left(x,t\right)u,\Im_x^*u_t\right)_{L^2(Q^\tau)}\\&+\left(\mathcal{L}u,x\Im_x^*u_t\right)_{L^2(Q^\tau)}-\left(\mathcal{L}u,\Im_x^*(\rho u_t)\right)_{L^2(Q^\tau)}.\end{aligned}$$

By applying the Cauchy inequality to the last three terms on the right-hand side of the inequality (15) and making use conditions C_1,combining with (15),we obtain

$$\begin{aligned}(16)\qquad &\left\|u(.,\tau)\right\|_{L^2(0,l)}^2+\left\|u_t(.,\tau)\right\|_{L^2(0,l)}^2\\&\qquad+\left\|\Im_x^*u_t(.,\tau)\right\|_{L^2(0,l)}^2\\&\le k\Big[\left\|f\right\|_{L^2(Q^\tau)}^2+\left\|\varphi_1\right\|_{L^2(0,l)}^2+\left\|\varphi_2\right\|_{L^2(0,l)}^2\\&\left\|u\right\|_{L^2(Q^\tau)}^2+\left\|u_t\right\|_{L^2(Q^\tau)}^2+\left\|\Im_x^*u_t\right\|_{L^2(Q^\tau)}^2\Big].\end{aligned}$$

where $k=\frac{\max\left(2,b_1,\beta+l^2,b_3^2+b_2,l^4\right)}{\min(1,b_0,\beta)}$.

Applying the Gronwall lemma to(16),and elimining the term $\left\|\Im_x^*u_t(.,\tau)\right\|_{L^2(0,l)}^2$ of the left-hand side of the inequality we obtain

$$\begin{aligned}(17)\qquad &\left\|u(.,\tau)\right\|_{L^2(0,l)}^2+\left\|u_t(.,\tau)\right\|_{L^2(0,l)}^2\\&\le k\exp(kT)\left(\left\|f\right\|_{L^2(Q^\tau)}^2+\left\|\varphi_1\right\|_{L^2(0,l)}^2+\left\|\varphi_2\right\|_{L^2(0,l)}^2\right).\end{aligned}$$

Since the left-hand side of (17)does not depend on τ,we take the supremum with τ from 0 to T, then the estimate (7) follows with $c = \sqrt{k}\exp(k\frac{T}{2})$.

5. Solvability of the problem

Proposition 1. The operator L acting from B to H have a closure.

Proof. (see [3])

Let be $\overline{L}$ the closure of L, $D(\overline{L})$ its domain .

Definition .The solution of $\overline{L}u = F$ for any $u \in D(\overline{L})$ is strong solution of problem(1)-(5). we take the limit in the inequality (7) ,we obtain$\|u\|_B \leq c \left\|\overline{L}u\right\|_H$,$\forall u {\in} D(\overline{L})$.

From the inequality we have:

Corollary 1: The strong solution of problem (1)-(5) when it exists, it's unique, and depends continuly of data f, φ_1, φ_2.

Corollary 2 :The set of values $R(\overline{L})$ of the operator $\overline{L}$ is equal to the closure $\overline{R(L)}$ of $R(L)$.

Theorem 2: If the conditions C_1-C_2 are satisfying ,then for any $\mathcal{F} = (f, \varphi_1, \varphi_2) \in H$, there exists a strong unique solution $u = \overline{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$ of the probleme (1)-(5) where the estimate $\|u\|_B \leq c\|\mathcal{F}\|_H$ is satisfying ,where c is a positive constant does not depends of u.

Proof: From(22) we conclude that the operator $\overline{L}$ acting from $D(\overline{L})$ in $R(\overline{L})$ have an inverse $\overline{L}^{-1}$, and from corollary 2, we conclude that the range $R(\overline{L})$ of the operator $\overline{L}$ is closed. Then we will be proove the density of the set $R(L)$ in the space H (i.e) $\overline{R(L)} = H$.

For this we need the following proposition :

Proposition 2:If,for all functions $u \in D_0(L)$, where

$$D_0(L) = \{u/u \in D(L) : l_1 u = l_2 u = 0\}$$

and for some function $\omega \in L^2(Q)$, we have

$$(\mathcal{L}u, \omega)_{L^2(Q)} = 0, \tag{18}$$

then ω vanishes almost everywhere in Q.

Proof of the proposition 2 : The relation (18) is given for all $u \in D_0(L)$,we can express it in a particular form .Let u_{tt} be a solution of :

$$b(\sigma, t)\left[x\Im_x^* u_{tt} - \Im_x^*(\rho u_{tt})\right] = h(x,t), \tag{19}$$

where σ is a constant in $(0, l)$ and $h(x,t) = \int_t^T \omega(x,\tau)d\tau$.
And let u be the fonction defined by:

$$u = \begin{cases} 0, & \text{si } 0 \le t \le s, \\ \int_s^t (t-\tau)u_{\tau\tau}d\tau, & \text{si } s \le t \le T. \end{cases} \tag{20}$$

(19) and (20) follows u is in $D_0(L)$ and:

$$\begin{aligned} \omega(x,t) &= \Im_x^{*^{-1}} h \\ &= -\left[b(\sigma,t)\left(x\Im_x^* u_{tt} - \Im_x^*(\rho u_{tt})\right)\right]_t \\ &= \left[b(\sigma,t)\Im_x^*(\rho - x)u_{tt}\right]_t . \end{aligned} \tag{21}$$

To continue the proof we need the following lemma :

Lemma 2. The function ω defined by (21),belongs to the space $L^2(Q)$.

Proof of lemma 2 :We start with the proof of this inequality $\|\Im_x^*(\rho - x)u_{tt}\|^2_{L^2(0,l)} \le \frac{l^4}{12}\|u_{tt}\|^2_{L^2(0,1)}$.

From this inequality and since the conditions C_1 are satisfied we conclude that $b_t(\sigma,t)\Im_x^*(\rho - x)u_{tt}$ belongs to $L^2(Q)$.

Because $\omega(x,t) = \left[b(\sigma,t)\Im_x^*(\rho - x)u_{tt}\right]_t =$
$= b_t(\sigma,t)\Im_x^*(\rho - x)u_{tt} + b(\sigma,t)\Im_x^*(\rho - x)u_{ttt}$, then we will

be prooved that: $b\,(\sigma,t)\,\Im_x^*(\rho-x)u_{ttt}\in L^2(Q)$.

For this we introduce the t-averaging opérators ρ_ε of the form

$$(\rho_\varepsilon f)(x,t)=\frac{1}{\varepsilon}\int_0^T\omega(\tfrac{t-s}{\varepsilon})f(x,s)ds,$$

where $\omega\in C_0^\infty(0,T),\omega\geq 0$,
$\int_{-\infty}^{+\infty}\omega(s)ds=1$, $\omega\equiv 0$ for $t\leq 0$ and $t\geq T$,
applying the operators ρ_ε and $\frac{\partial}{\partial t}$ to the equation

$$-b\,(\sigma,t)\,\Im_x^*(\rho-x)u_{tt}=h(x,t),$$

we obtain

$$\frac{\partial}{\partial t}\left(-b\,(\sigma,t)\,\Im_x^*(\rho-x)u_{tt}\right)=$$
$$\frac{\partial}{\partial t}\left[-b\,(\sigma,t)\,\Im_x^*(\rho-x)u_{tt}+\rho_\varepsilon\left(b\,(\sigma,t)\,\Im_x^*(\rho-x)u_{tt}\right)\right]-\frac{\partial}{\partial t}\rho_\varepsilon h.$$

Then
$$\left\|b\,(\sigma,t)\,\Im_x^*(\rho-x)u_{tt}\right\|_{L^2(Q)}^2$$
$$\leq 2\left\|\frac{\partial}{\partial t}\left[b\,(\sigma,t)\,\Im_x^*(\rho-x)u_{tt}-\rho_\varepsilon\left(b\,(\sigma,t)\,\Im_x^*(\rho-x)u_{tt}\right)\right]\right\|_{L^2(Q)}^2$$
$$+2\left\|\frac{\partial}{\partial t}\rho_\varepsilon h\right\|_{L^2(Q)}^2.$$

Since $\rho_\varepsilon f\underset{\varepsilon\to 0}{\longrightarrow}f$, and $\frac{\partial}{\partial t}\left(b\,(\sigma,t)\,\Im_x^*(\rho-x)u_{tt}\right)$ is bounded in $L^2(Q)$, then $\omega\in L^2(Q)$.

Now we return to the 2^{nd} proposition ,we remplace ω in (18) by its representation given by (21) we have:

$$\left(u_{tt},\left[b\,(\sigma,t)\,\Im_x^*(\rho-x)u_{tt}\right]_t\right)_{L^2(Q)} \tag{22}$$
$$=\left(\left(b\,(x,t)\,u_x\right)_x,\left[b\,(\sigma,t)\,\Im_x^*(\rho-x)u_{tt}\right]_t\right)_{L^2(Q)}+$$
$$+\beta\left(u_{ttxx},\left[b\,(\sigma,t)\,\Im_x^*(\rho-x)u_{tt}\right]_t\right)_{L^2(Q)}.$$

Making use conditions(3)-(5),and from the particular forme of u given by (19) and (20), the equality (22) can be simplified.For this integrating by parts each term of the equality on the sup-domain $Q_s = (0,l) \times (s,T)$ where $0 \leq s \leq T$

$$(23) \qquad \left(u_{tt}, \left[b\left(\sigma,t\right)\Im_x^*(\rho-x)u_{tt}\right]_t\right)_{L^2(Q)}$$
$$= \frac{1}{2}\left\|\sqrt{b\left(\sigma,s\right)}\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)} - \frac{1}{2}\left\|\sqrt{b_t\left(\sigma,.\right)}\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)},$$

$$(24) \qquad \left(\left(b\left(x,t\right)u_x\right)_x, \left[b\left(\sigma,t\right)\Im_x^*(\rho-x)u_{tt}\right]_t\right)_{L^2(Q)}$$
$$= -\frac{1}{2}\left\|\sqrt{b\left(.,T\right)b\left(\sigma,T\right)}u_t(.,T)\right\|^2_{L^2(0,l)}$$
$$+\frac{1}{2}\int_{Q_s}\left[3b_t\left(x,t\right)b\left(\sigma,t\right)+b\left(x,t\right)b_t\left(\sigma,t\right)\right]\left(u_t\right)^2dxdt$$
$$-\int_0^l b_t\left(x,T\right)b\left(\sigma,T\right)u\left(x,T\right)u_t\left(x,T\right)dx$$
$$+\int_{Q_s}\left[b_{tt}\left(x,t\right)b\left(\sigma,t\right)+b_t\left(x,t\right)b_t\left(\sigma,t\right)\right]uu_tdxdt$$
$$+\int_{Q_s}\left[b_x\left(x,t\right)u_t+b_{xt}\left(x,t\right)u\right]b\left(\sigma,t\right)\Im_x^* u_{tt}dxdt,$$

$$(25) \qquad \beta\left(u_{ttxx}, \left[b\left(\sigma,t\right)\Im_x^*(\rho-x)u_{tt}\right]_t\right)_{L^2(Q)}$$
$$= \frac{\beta}{2}\left\|\sqrt{b_t\left(\sigma,.\right)}u_{tt}\right\|^2_{L^2(Q_s)} - \frac{\beta}{2}\left\|\sqrt{b\left(\sigma,s\right)}u_{tt}\left(.,s\right)\right\|^2_{L^2(0,l)}.$$

Substitution of(23)-(25)into (22) gives

$$
\begin{aligned}
(26)\qquad & \frac{1}{2}\left\|\sqrt{b(\sigma,s)}\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)} \\
& +\frac{1}{2}\left\|\sqrt{b(.,T)\,b(\sigma,T)}u_t(.,T)\right\|^2_{L^2(0,l)} \\
& +\frac{\beta}{2}\left\|\sqrt{b(\sigma,s)}u_{tt}(.,s)\right\|^2_{L^2(0,l)} \\
= & \frac{1}{2}\left\|\sqrt{b_t(\sigma,s)}\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)}+\frac{\beta}{2}\left\|\sqrt{b_t(\sigma,.)}u_{tt}\right\|^2_{L^2(Q_s)} \\
& +\frac{1}{2}\int_{Q_s}\left[3b_t(x,t)\,b(\sigma,t)+b(x,t)\,b_t(\sigma,t)\right](u_t)^2\,dxdt \\
& -\int_0^l b_t(x,T)\,b(\sigma,T)\,u(x,T)\,u_t(x,T)\,dx \\
& +\int_{Q_s}\left[b_{tt}(x,t)\,b(\sigma,t)+b_t(x,t)\,b_t(\sigma,t)\right]uu_t dxdt \\
& +\int_{Q_s}\left[b_x(x,t)\,u_t+b_{xt}(x,t)\,u\right]b(\sigma,t)\,\Im_x^* u_{tt} dxdt\ .
\end{aligned}
$$

By applying the Cauchy inequality and Cauchy inequality with ε to estimate the last three terms on the right-hand side of the inequality (26) and making use conditions $\mathrm{C}_1 - C_2$, combining the estimates and(26)taking into account that $\varepsilon = \frac{b_0^2}{2b_1^2}$ we obtain

$$\frac{b_0}{2}\left[\left\|\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)} + \frac{b_0}{2}\left\|u_t(.,T)\right\|^2_{L^2(0,l)} + \right. \tag{27}$$
$$\left. + \beta\left\|u_{tt}\left(.,s\right)\right\|^2_{L^2(0,l)}\right]$$
$$\leq \left(b_1^2 + \frac{b_2}{2}\right)\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)} + \frac{\beta b_2}{2}\left\|u_{tt}\right\|^2_{L^2(Q_s)} +$$
$$+ \frac{b_2^2 + b_1^2 + 4b_1 b_2 + b_3^2}{2}\left\|u_t\right\|^2_{L^2(Q_s)} +$$
$$+ \frac{b_2^2 + b_4^2 + b_5^2}{2}\left\|u\right\|^2_{L^2(Q_s)} + \frac{b_1^2 b_2^2}{b_0^2}\left\|u\left(.,T\right)\right\|^2_{L^2(0,l)}$$

By virtue of the elementary inequality

$$\frac{b_1^2 b_2^2}{b_0^2}\left\|u\left(.,T\right)\right\|^2_{L^2(0,l)} \leq \frac{b_1^2 b_2^2}{b_0^2}\left\|u\right\|^2_{L^2(Q_s)} + \frac{b_1^2 b_2^2}{b_0^2}\left\|u_t\right\|^2_{L^2(Q_s)}, \tag{28}$$

we estimate the last term of the right-hand side of the inequality(27),we obtain

$$\left\|\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)} + \tag{29}$$
$$+ \frac{b_0}{2}\left\|u_t(.,T)\right\|^2_{L^2(0,l)} + \beta\left\|u_{tt}\left(.,s\right)\right\|^2_{L^2(0,l)}$$
$$\leq \left(\frac{2b_1^2 + b_2}{b_0}\right)\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)} + \frac{\beta b_2}{b_0}\left\|u_{tt}\right\|^2_{L^2(Q_s)}$$
$$+ \frac{\frac{2b_1^2 b_2^2}{b_0^2} + 2b_1 b_2 + \left(b_2 + b_1\right)^2 + b_3^2}{b_0}\left\|u_t\right\|^2_{L^2(Q_s)}$$
$$+ \frac{b_0^2\left(b_2^2 + b_4^2 + b_5^2\right) + 2b_1^2 b_2^2}{b_0^3}\left\|u\right\|^2_{L^2(Q_s)}$$

For estimate the last term of the right-hand side of the inequality(29),we will be proove the inequality $\left\|u\right\|^2_{L^2(Q_s)} \leq 24T^2\left\|u_t\right\|^2_{L^2(Q_s)}$, combining the last inequality and (29) we get

$$\begin{aligned} (30) \qquad & \left\|\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)} + \left\|u_{tt}\left(.,s\right)\right\|^2_{L^2(0,l)} + \\ & + \left\|u_t(.,T)\right\|^2_{L^2(0,l)} \\ & \leq k\left[\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)} + \left\|u_{tt}\right\|^2_{L^2(Q_s)} + \left\|u_t\right\|^2_{L^2(Q_s)}\right], \end{aligned}$$

where $k = \frac{\max\left(\beta b_2, \left(2b_1^2 + b_2\right), b_0 k(b_i,T)\right)}{b_0 \min\left(1, \beta, \frac{b_0}{2}\right)}$.

$k\left(b_i, T\right) = \frac{2b_1^2 b_2^2 + b_0^2\left[2b_1 b_2 + \left(b_2 + b_1\right)^2 + b_3^2\right] + 24T^2\left[b_0^2\left(b_2^2 + b_4^2 + b_5^2\right) + 2b_1^2 b_2^2\right]}{b_0^3}$.

To continue,we introduce the new function $v(x,t) = \int_t^T u_{\tau\tau} d\tau$, then $u_t(x,t) = v(x,s) - v(x,t)$, and $u_t(x,T) = v(x,s)$.

The inequality (30) it be

$$\begin{aligned} (31) \qquad & \left\|\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)} + \left\|u_{tt}\left(.,s\right)\right\|^2_{L^2(0,l)} + \\ & + \left(1 - 2k(T-s)\right)\left\|v(.,s)\right\|^2_{L^2(0,l)} \\ & \leq 2k\left(\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)} + \left\|u_{tt}\right\|^2_{L^2(Q_s)} + \left\|v\right\|^2_{L^2(Q_s)}\right). \end{aligned}$$

If $s_0 > 0$ satisfies $\left(1 - 2k(T - s_0)\right) = \frac{1}{2}$,then the inequality (31) implies

$$\begin{aligned} (32) \qquad & \left\|\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)} + \left\|u_{tt}\left(.,s\right)\right\|^2_{L^2(0,l)} \\ & + \left\|v(.,s)\right\|^2_{L^2(0,l)} \\ & \leq 4k\left(\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)} + \left\|u_{tt}\right\|^2_{L^2(Q_s)} + \left\|v\right\|^2_{L^2(Q_s)}\right), \end{aligned}$$

for all $s \in [T - s_0, T]$.We denote

$$Y(s) = \left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)} + \left\|u_{tt}\right\|^2_{L^2(Q_s)} + \left\|v\right\|^2_{L^2(Q_s)}.$$

We get:

$$Y'(s) = -\left\|\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)} - \left\|u_{tt}\left(.,s\right)\right\|^2_{L^2(0,l)} - \left\|v(.,s)\right\|^2_{L^2(0,l)}.$$

Then and from (32) we obtain $-Y'(s) \leq 4kY(s)$.
Then $-\frac{\partial}{\partial s}(Y(s)\exp(4ks)) \leq 0$.

Integrating this inequality on (s,T) and taking into account that $Y(T)=0$, we obtain $Y(s)\exp(4ks) \leq 0$.
Then $Y(s)=0$ for all $s \in [T-s_0, T]$ .Then $\omega = 0$ almost everywhere in Q_{T-s_0}, proceding in this way step by step,we proove that $\omega = 0$ almost everywhere in Q.

This achieves the proof of proposition.Now we return to proove the théorème.We will be proove that $\overline{R(L)} = H$.
Since H is a Hilbert space ,the equality $\overline{R(L)} = H$ is true,if from
(33)
$(Lu, W)_H = (\mathcal{L}u, \omega)_{L^2(Q)} + (l_1u, \omega_1)_{L^2(0,l)} + (l_2u, \omega_2)_{L^2(0,l)} = 0,$

where $W = (\omega, \omega_1, \omega_2) \in R(L)^{\perp}$,we get $\omega \equiv 0$,$\omega_1 \equiv 0$ and $\omega_2 \equiv 0$ in Q,for any element of $D_0(L)$.
From (33) we obtain $\forall u \in D_0(L), (\mathcal{L}u, \omega)_{L^2(Q)} = 0$.Then by virtue of the 2^{nd} proposition , we conclude that $\omega \equiv 0$.
Then for (33),we obtain $(l_1u, \omega_1)_{L^2(0,l)} + (l_2u, \omega_2)_{L^2(0,l)}$ =0.

Since the quantities l_1u and l_2u can vanish independently and the ranges of the trace operators l_1 and l_2 are dense in the Hilbert space $L^2(0,l)$,then $\omega_1 = \omega_2 = 0$ .Thus to conclude that $W = 0$.

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